is due to the interchange of the growth and diminution of $\omega(\tau)$ (Fig. 2c). For comparison, let us show that in a similar problem examined in [2] and in a number of problems on Euler elastica, the modes with an inflection are known to be unstable.

The displacements are determined as the sum of the function $v(\tau)$ found earlier and a second degree polynomial in τ in the problem of equilibrium of a string without an external load under the effect of a curved magnet. Hence, only positive forms without inflection points are obtained, as it should be also in the case of a load which does not change sign for any displacements.

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ON GENERAL RELATIONSHIPS OF THE THEORY OF IDEAL PLASTICITY AND THE STATICS OF A FRIABLE MEDIUM

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D. D. IVLEV
(Moscow)

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General relationships of the theory of ideal plasticity and the statics of a friable medium for Tresca plasticity condition and its extensions, on the basis of determining the dissipation function, are considered. The work is related to the investigations in [1, 2].

1. Under the Tresca plasticity condition, the dissipation function is

$$D = 2k \mid \varepsilon_i \mid_{\max}, \qquad k = \text{const} \tag{1.1}$$

where $\varepsilon_{i\max}$ is the maximum principal strain rate component. For definiteness, we later assume $\varepsilon_i = \varepsilon_3$; we shall consider the material incompressible. Let us write the initial functional to determine the associated loading law as

$$D = 2k\,\varepsilon_3(\varepsilon_{ij}) + \mu(\varepsilon_x + \varepsilon_y + \varepsilon_z) \tag{1.2}$$

where ε_{ij} are the components of the strain rate tensor, μ is a Lagrange multiplier. It is necessary to know the expression $\varepsilon_3 = \varepsilon_3(\varepsilon_{ij})$. Let n_i denote the direction cosines of the third principal direction in a Cartesian coordinate system x_i. Then $n_i\varepsilon_3 = \varepsilon_{ij}n_j$. Hence, the known formula follows

$$\varepsilon_3 = \varepsilon_{ij}\, n_i n_j \tag{1.3}$$

Using (1.3), it is necessary to take into account that $n_i = n_i(\varepsilon_{ij})$ since the orientation of the principal directions change when the components of the strain rate tensor change. Taking account of (1.2), (1.3), in conformity with the associated loading law we obtain

$$\sigma_{mn} = \frac{\partial D}{\partial \varepsilon_{mn}} = \frac{\partial}{\partial \varepsilon_{mn}} [\varepsilon_{ij} n_i (\varepsilon_{kl}) n_j (\varepsilon_{pq})] + \mu \frac{\partial}{\partial \varepsilon_{mn}} (\varepsilon_{ij} \delta_{ij}) \tag{1.4}$$

We show that

$$\frac{\partial}{\partial \varepsilon_{mn}} [\varepsilon_{ij} n_i (\varepsilon_{kl}) n_j (\varepsilon_{pq})] = n_m n_n \tag{1.5}$$

For example, let $\varepsilon_{mn} = \varepsilon_x$, then

$$\frac{\partial}{\partial \varepsilon_x} (\varepsilon_{ij} n_i n_j) = n_1^2 + 2\varepsilon_{ij} n_i \frac{\partial n_j}{\partial \varepsilon_x} = n_1^2 + 2\varepsilon_s \left(n_i \frac{\partial n_i}{\partial \varepsilon_x} \right) \tag{1.6}$$

Since $n_1^2 + n_2^2 + n_3^2 = 1$, then the last expression in parantheses in (1.6) equals zero. The assertion (1.5) is proved analogously in the general case. According to (1.4), (1.5) we have

$$\sigma_x = \mu + 2kn_1^2, \ldots \qquad \tau_{xy} = 2kn_1 n_2, \ldots \tag{1.7}$$

The expressions not written down in (1.7) are obtained by cyclic permutation of the subscripts. According to (1.7)

$$\mu = \sigma - {}^2/_3 k, \qquad \sigma = {}^1/_3 \sigma_{ij} \delta_{ij} \tag{1.8}$$

The relationships (1.7), (1.8) determine the plasticity conditions corresponding to the edge of the Tresca prism, known as the "total plasticity condition". For the faces of the Tresca prism $\sigma_1 - \sigma_2 = 2k$ $(\sigma_2 \leqslant \sigma_3 \leqslant \sigma_1)$. In this case, from the associated flow law $\varepsilon_1 = \lambda$, $\varepsilon_2 = -\lambda$, $\varepsilon_3 = 0$. The dissipation function is

$$D = \sigma_1 \varepsilon_1 + \sigma_2 \varepsilon_2 + \sigma_3 \varepsilon_3 = 2k\lambda = 2k\varepsilon_1$$

We shall proceed from the dissipation function $D = 2k\varepsilon_1$ under the conditions $\varepsilon_1 + \varepsilon_2 = 0$, $\varepsilon_3 = 0$. The initial functional is of the form

$$D = 2k\varepsilon_1 + \mu_1 (\varepsilon_1 + \varepsilon_2) + \mu_2 \varepsilon_3 \tag{1.9}$$

where μ_1, μ_2 are Lagrange factors. Let l_i, m_i be the direction cosines of the principal directions ε_1, ε_2 from (1.11). According to (1.4), (1.5), we obtain

$$\sigma_x = 2kl_1^2 + \mu_2 (l_1^2 + m_1^2) + \mu_2 n_1^2, \ldots$$
$$\tau_{xy} = 2kl_1 l_2 + \mu_1 (l_1 l_2 + m_1 m_2) + \mu_2 n_1 n_2, \ldots \tag{1.10}$$

There follows from (1.12)

$$\sigma = 2/3\, k + \mu_1 + \mu_2 \tag{1.11}$$

After eliminating the quantities μ_1, μ_2 from (1.10) as is done in [1], we obtain the plasticity condition in the Levy form; the corresponding faces of the Tresca prism are

$$4 (q + k^2)(q + 4k^2)^2 + 27 r^2 = 0$$
$$q = \sigma_{ij}' \sigma_{ij}', \qquad r = \sigma_{ij}' \sigma_{jk}' \sigma_{ki}' \tag{1.12}$$

the primes are ascribed to the deviator components.

2. Let us write the fundamental limit condition of the statics of a friable medium as

$$\max |\tau_n| = k + \sigma_n \operatorname{tg} \rho, \quad k, \rho - \text{const} \tag{2.1}$$

where τ_n, σ_n are the shear and normal stresses. The relationship (2.1) determines a Coulomb prism in the space of principal stresses, for which the equation of the edge can be written as

$$(\sigma_3 - \sigma_1) - (\sigma_1 + \sigma_3) \sin \rho = 2k \cos \rho$$
$$(\sigma_3 - \sigma_2) - (\sigma_2 + \sigma_3) \sin \rho = 2k \cos \rho \tag{2.2}$$

In conformity with the generalized associated flow law

$$\varepsilon_2 = \lambda_2(-1 - \sin\rho), \quad \varepsilon_3 = \lambda_2(-1 - \sin\rho) \tag{2.3}$$
$$\varepsilon_3 = \lambda_1(1 - \sin\rho) + \lambda_2(1 - \sin\rho)$$

From (2.3) we obtain

$$D = \sigma_i\varepsilon_i = 2k\cos\rho\,(\lambda_1 + \lambda_2) \tag{2.4}$$

$$\lambda_1 + \lambda_2 = -\frac{1}{2\sin\rho}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3), \quad \lambda_1 + \lambda_2 = {}^1/_2\,[\varepsilon_3 - (\varepsilon_1 + \varepsilon_2)] \tag{2.5}$$

Therefore, the dilatancy dependence holds

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + [\varepsilon_3 - (\varepsilon_1 + \varepsilon_2)]\sin\rho = 0 \tag{2.6}$$

In determining the relationships of the statics of a friable medium by proceeding from the definition of the dissipative function, the presence of the dilatancy dependence (2.6) should be postulated. The initial functional should be taken in one of the equivalent forms

$$D = -\frac{k\cos\rho}{\sin\rho}(\varepsilon_x + \varepsilon_y + \varepsilon_z) + \mu_1\,]\varepsilon_x + \varepsilon_y + \varepsilon_z + (\varepsilon_3 - \varepsilon_1 - \varepsilon_2)\sin\rho] \tag{2.7}$$

$$D = k\cos\rho\,(\varepsilon_3 - \varepsilon_1 - \varepsilon_2) + \mu_2\,[\varepsilon_x + \varepsilon_y + \varepsilon_z + (\varepsilon_3 - \varepsilon_1 - \varepsilon_2)\sin\rho \tag{2.8}$$

where μ_1, μ_2 are Lagrange multipliers. Let us proceed from (2.7). Transforming (2.7) into the form

$$D = -\frac{k\cos\rho}{\sin\rho}(\varepsilon_x + \varepsilon_y + \varepsilon_z) + \mu_1\,[(\varepsilon_x + \varepsilon_y + \varepsilon_z)(1 - \sin\rho) + \varepsilon_2\sin\rho] \tag{2.9}$$

and taking account of (1.7), we obtain

$$\sigma_x = -\frac{k\cos\rho}{\sin\rho} + \mu\,[(1 - \sin\rho) + 2n_1^2\sin\rho], \ldots, \tau_{xy} = 2\mu n_1 n_2 \sin\rho, \ldots \tag{2.10}$$

It follows from (2.10) that

$$\sigma = -\frac{k\cos\rho}{\sin\rho} + \mu\left[1 - \frac{1}{3}\sin\rho\right] \tag{2.11}$$

Relationships determining the plasticity conditions corresponding to an edge of the Coulomb prism, considered in [1], follow from (2.10), (2.11). The general case of the dependence $\max\{|\tau_n| - f(\sigma_n)\} = 0$ can be considered analogously.

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